Esercizi su integrali tripli: cambiamento di variabili

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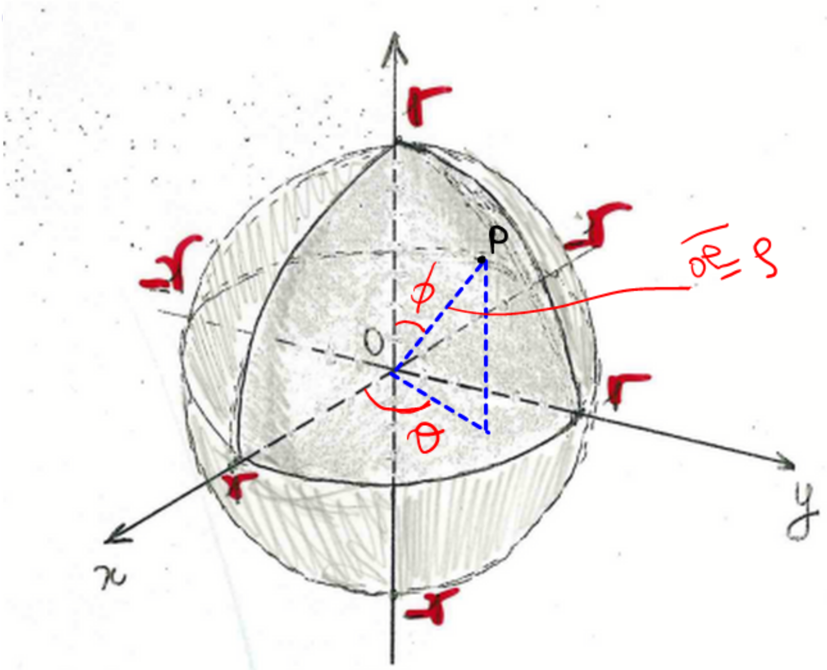
Analisi II


r
P
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x
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Coordinate sferiche

$$\begin{cases} x = \rho\cos(\theta)\sin(\phi) \\ y = \rho\sin(\theta)\sin(\phi) \\ z = \rho\cos(\phi) \end{cases} \qquad \rho \in [0,+\infty),\ \phi \in [0,\pi],\ \theta \in [0,2\pi]$$

Si calcola

$$J = \rho^2 \sin(\phi) \quad \Rightarrow \quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \ \to \ \rho^2 \sin(\phi)\ \mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Quindi

$$\iiint_T f(x,y,z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \iiint_{\widetilde{T}} f(\rho\cos(\theta)\sin(\phi), \rho\sin(\theta)\sin(\phi), \rho\cos(\phi))\rho^2\sin(\phi)\ \mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

- Il passaggio alle coordinate sferiche è conveniente se

$$\text{nell'espressione di } \underline{f}, \ \underline{\text{o}} \text{ nell'espressione di } \underline{T},$$

compaiono espressioni con

$$x^2 + y^2 + z^2 \quad (\text{in coordinate sferiche } \rightarrow \ \rho^2)$$

Es. 1.

$$I = \iiint_T \sqrt{|z|}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

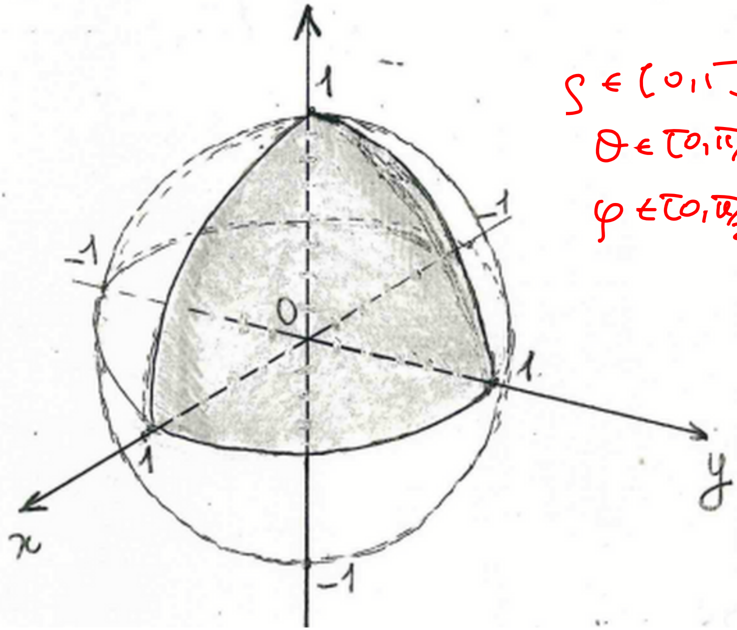
ove T è la porzione di sfera (solida) nel primo ottante

$$T = \left\{(x, y, z) \in \mathbb{R}^3 \, : \, x \geq 0, \; y \geq 0, \; z \geq 0, \; x^2 + y^2 + z^2 \leq 1\right\}.$$

La simmetria sferica del dominio di integrazione suggerisce l'uso delle coordinate sferiche:

$$\left.\begin{array}{l} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array}\right\} \longrightarrow \begin{array}{l} \vartheta \in [0, \pi/2] \\ \varphi \in [0, \pi/2] \end{array}$$

$$x^2 + y^2 + z^2 \leq 1 \quad \Updownarrow \quad \rho \leq 1$$



$\rho \in [0,1]$

$\theta \in [0, \pi/2]$

$\varphi \in [0, \pi/2]$

$$I = \iiint_{[0,1]\times[0,\pi/2]\times[0,\pi/2]} \underbrace{\rho^{1/2}|\cos(\phi)|^{1/2}}_{=\sqrt{|z|}}\underbrace{\rho^2\sin(\phi)\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta}_{=dx\,dy\,dz}$$

$$= \iiint_{[0,1]\times[0,\pi/2]\times[0,\pi/2]} \rho^{5/2}\cos(\varphi)^{1/2}\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

$$= \left(\int_0^1 \rho^{5/2}\,d\rho\right)\cdot\left(\int_0^{\pi/2}\cos(\varphi)^{1/2}\sin(\varphi)\,d\varphi\right)\cdot$$

$$\cdot\left(\int_0^{\pi/2} 1\,d\theta\right)$$

$$= \frac{\pi}{2}\left[\frac{2}{7}\rho^{7/2}\right]_0^1\cdot\left[-\frac{2}{3}\cos(\varphi)^{3/2}\right]_0^{\pi/2}$$

$$= \frac{2}{21}\pi$$

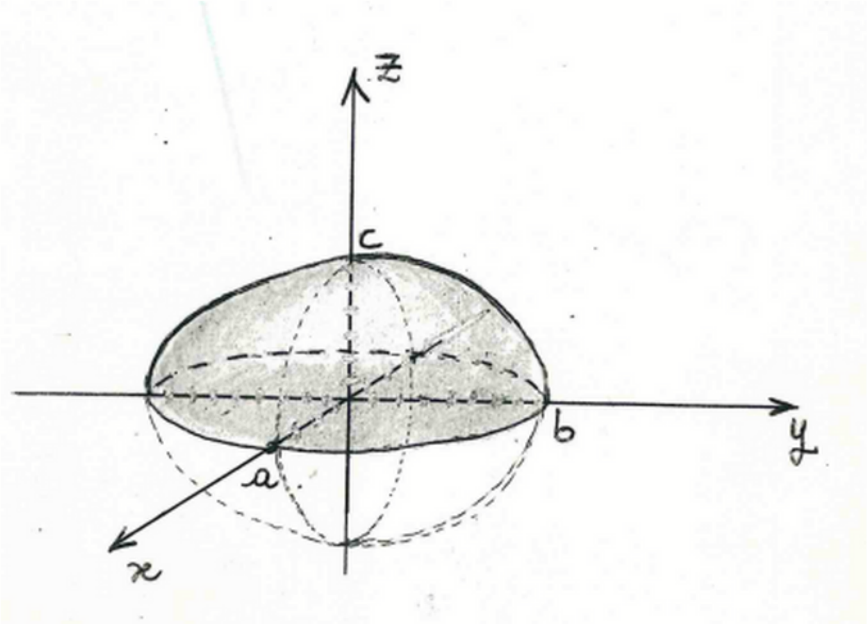
Es. 2.

$$I = \iiint_{T_+} z \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

con

$$T_+ = T \cap \{z \geq 0\} = \left\{ (x, y, z) \in \mathbb{R}^3 \ : \ z \geq 0, \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\}$$

con $a, \ b, \ c, \ R > 0$.


z
c
b
y
a
x

La geometria del dominio suggerisce il passaggio alle *coordinate sferiche generalizzate*

$$\begin{cases} x = a\rho\cos(\theta)\sin(\phi) \\ y = b\rho\sin(\theta)\sin(\phi) \\ z = c\rho\cos(\phi) \end{cases}$$

$$T \ \to \tilde{T} : \ \rho \in [0, R], \ \phi \in [0, \pi], \ \theta \in [0, 2\pi]$$

Si calcola

$$J = abc\rho^2 \sin(\phi) \quad \Rightarrow \quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \ \to \ abc\rho^2 \sin(\phi)\ \mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta$$

$$I = \iiint_{[0,1]\times[0,\pi/2]\times[0,2\pi]} c\,\rho\cos(\varphi)\; abc\,\rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= abc^2 \left(\int_0^1 \rho^3\, d\rho\right)\cdot\left(\int_0^{\pi/2} \cos(\varphi)\sin(\varphi)\, d\varphi\right)\cdot\left(\int_0^{2\pi} 1\, d\theta\right)$$

$$= 2\pi abc^2 \cdot \frac{1}{4}\cdot\left[\frac{\sin^2(\varphi)}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi abc^2}{4}$$

Esercizio assegnato

Calcolare il volume di T

$$T = \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq R^2 \right\}$$

con a, b, c, $R > 0$.

Si ha

$$\mathrm{vol}(T) = \iiint_T 1 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z.$$

$$\mathrm{vol}(T) = \frac{4}{3} \pi a b c R^3$$

N.B. Per $a=b=c=1$ l'ellissoide si riduce a una superficie sferica di raggio R. Ritrovo la formula per il volume della sfera.

Es. 3.

Calcolare il volume di

$$T = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \le y\}$$

In T compare $x^2 + y^2 + z^2$ quindi uso le coordinate sferiche!

$$\begin{cases} x = \rho \cos(\theta) \sin(\phi) \\ y = \rho \sin(\theta) \sin(\phi) \\ z = \rho \cos(\phi) \end{cases}$$

Quale è la trasformazione $T \to \tilde{T}$?? Impongo

$x^2+y^2+z^2 \le y$ in coord. sferiche

$\Updownarrow$

$\rho^2 \le \rho \sin(\varphi) \sin(\theta)$

$\Leftrightarrow \quad \rho(\rho - \sin(\varphi)\sin(\vartheta)) \leq 0$

$\Leftrightarrow \quad \boxed{\rho \leq \sin(\varphi)\sin(\vartheta)}$

$\Downarrow$

$\boxed{\rho \geq 0}$

da qui leggo che

$\sin(\varphi)\cdot\sin(\vartheta) \geq 0.$

Siccome $\varphi \in [0,\pi]$, $\sin(\varphi) \geq 0.$

Allora trovo il vincolo $\sin(\vartheta) \geq 0$,

da cui $\boxed{\vartheta \in [0,\pi]}$

Quindi $\tilde{T}$: $\begin{cases} 0 \le \rho \le \sin(\theta)\sin(\varphi) \\ \theta \in [0,\pi] \\ \varphi \in [0,\pi] \end{cases}$

è un dominio "normale rispetto al piano $\varphi\theta$"

Integro per fili:

$$\mathrm{vol}(T) = \iiint_T 1 \, dx\,dy\,dz$$

$$= \iiint_{\tilde{T}} 1 \, \rho^2 \sin(\varphi) \, d\rho \, d\varphi \, d\theta$$

$$= \iint_{[0,\pi]\times[0,\pi]} \left(\int_0^{\sin(\theta)\sin(\varphi)} \rho^2 \sin(\varphi)\, d\rho \right) d\theta\, d\varphi$$

$$= \iint_{[0,\pi]\times[0,\pi]} \sin(\varphi) \left[\frac{\rho^3}{3} \right]_0^{\sin(\theta)\sin(\varphi)} d\theta\, d\varphi$$

$$= \frac{1}{3} \iint_{[0,\pi]\times[0,\pi]} \sin^3(\theta)\, \sin^4(\varphi)\, d\theta\, d\varphi$$

$$= \frac{1}{3} I_1 \cdot I_2$$

$$I_1 = \int_0^{\pi} \underbrace{\sin^3(\theta)}_{\substack{= \\ \sin\theta \cdot \sin^2(\theta) \\ = \\ \sin\theta\,(1-\cos^2(\theta))}} d\theta$$

$$= \int_0^{\pi} \sin(\theta)\, d\theta - \int_0^{\pi} \cos^2(\theta)\sin(\theta)\, d\theta$$

$$= \Big[-\cos(\theta)\Big]_0^{\pi} + \left[\frac{\cos^3(\theta)}{3}\right]_0^{\pi}$$

$$= \frac{4}{3}$$

$$I_2 = \int_0^{\pi} \underbrace{\sin^4(\varphi)}_{\substack{= \\ \sin^2(\varphi)\cdot\sin^2(\varphi) \\ = \\ \sin^2(\varphi)\,(1-\cos^2(\varphi))}} \, d\varphi$$

$$= \underbrace{\int_0^{\pi} \sin^2(\varphi)\, d\varphi}_{= I_{2,1}} \underbrace{- \int_0^{\pi} \sin^2(\varphi)\cos^2(\varphi)\, d\varphi}_{= I_{2,2}}$$

$$I_{2,1} = \left[\varphi - \frac{\sin(\varphi)\cos(\varphi)}{2} \right]_0^{\pi} = \pi/2$$

$$I_{2,2} = \int_0^{\pi} \sin(\varphi)\,(-\sin(\varphi))\cdot\cos^2(\varphi)\, d\varphi$$

$$= \int_0^{\pi} \sin(\varphi) \cdot \frac{d}{d\varphi}\left(\frac{\cos^3(\varphi)}{3}\right) d\varphi$$

integro per parti

$$= -\int_0^{\pi} \cos(\varphi) \cdot \frac{\cos^3(\varphi)}{3} d\varphi + \underbrace{\left[\sin(\varphi) \frac{\cos^3(\varphi)}{3} \right]_0^{\pi}}_{=0}$$

$$= -\frac{1}{3} \int_0^{\pi} \underbrace{\cos^4(\varphi)}_{= (1-\sin^2(\varphi))^2 = 1+\sin^4(\varphi)-2\sin^2(\varphi)} d\varphi$$

$$\Rightarrow I_{2,2} = -\frac{1}{3}\left[\cancel{\int_0^{\pi} 1\, d\varphi} + \int_0^{\pi} \sin^4(\varphi)\, d\varphi \right.$$
$$\left. - 2\cancel{\int_0^{\pi} \sin^2(\varphi)\, d\varphi}\right]$$

Perché

$$\int_0^{\pi} \sin^2(\varphi) = \frac{\pi}{2}$$

$$\Rightarrow I_{2,2} = -\frac{1}{3}\int_0^{\pi} \sin^4(\varphi)\, d\varphi$$

Rimettendo tutto insieme, ho

$$\int_0^{\pi} \sin^4(\varphi)\, d\varphi = \frac{\pi}{2} - \frac{1}{3}\int_0^{\pi} \sin^4(\varphi)\, d\varphi$$

$$\Rightarrow \frac{4}{3}\int_0^{\pi} \sin^4(\varphi)\, d\varphi = \pi/2$$

$$\Rightarrow I_2 = 3/8\,\pi$$

$$\Rightarrow vol(T) = \frac{1}{3}\cdot\frac{4}{3}\cdot\frac{3}{8}\pi = \pi/6$$

Es. 4.

$$I = \iiint_T \ln(x^2 + y^2 + z^2) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

ove $T = B \cap C$, con

$$B = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 \leq 1\right\} \quad \text{sfera unitaria}$$

$$C = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leq z^2, \ z \geq 0\right\}$$

porzione di cono solido nel semispazio $\{z \geq 0\}$

- L'espressione $x^2 + y^2 + z^2$ compare nella funz. integranda e in B, quindi

passo alle coordinate sferiche!

- Come $T \to \tilde{T}$??

. $x^2+y^2+z^2 \leq 1 \iff \rho \in [0,1]$

. $z \geq 0 \iff \phi \in [0, \pi/2]$

. $x^2+y^2 \leq z^2 \iff$

$$\rho^2\cos^2(\theta)\sin^2(\varphi)+\rho^2\sin^2(\theta)\sin^2(\varphi) \leq \rho^2\cos^2(\varphi)$$

$$\iff \rho^2\sin^2(\varphi) \leq \rho^2\cos^2(\varphi)$$

$$\iff \sin^2(\varphi)-\cos^2(\varphi) \leq 0$$

$$\iff \underbrace{[\sin(\varphi)+\cos(\varphi)]}_{\geq 0 \text{ perché } \varphi\in[0,\pi/2]}[\sin\varphi-\cos(\varphi)] \leq 0$$

$\Leftrightarrow \sin(\varphi) \leq \cos(\varphi)$

$\Leftrightarrow \varphi \in [0, \pi/4]$.

Non ci sono vincoli su θ!

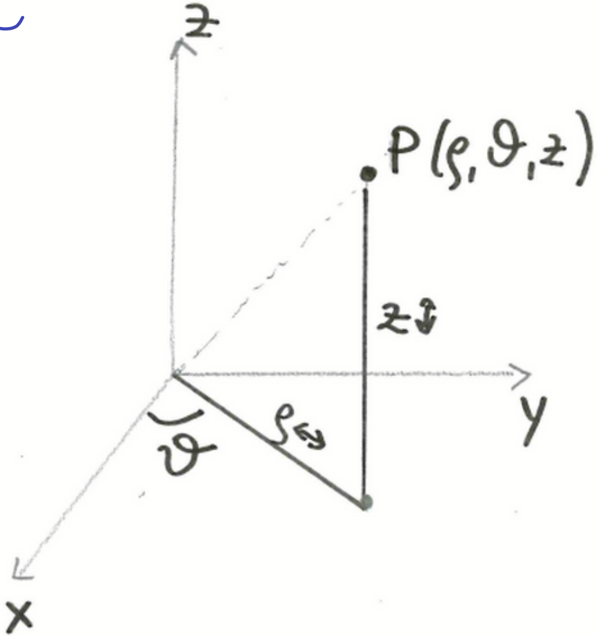
$$\tilde{T}: \quad \rho \in [0,1], \ \theta \in [0, 2\pi], \ \in [0, \pi/4]$$

$$I = \iiint_{[0,1]\times[0,2\pi]\times[0,\pi/4]} \ln(\rho^2) \cdot \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \left[\int_0^{\pi/4} \sin(\phi)\, d\phi\right] \cdot \left[\int_0^{2\pi} 1 \, d\theta\right] \cdot \left[\int_0^1 \rho^2 \ln(\rho^2)\, d\rho\right]$$

$$= \ldots\ldots = \boxed{-\frac{2\sqrt{2}}{9}\pi\left(\sqrt{2}-1\right)}$$

Coordinate
cilindriche



Coordinate polari cilindriche

$$\begin{cases} x = \rho\cos(\theta) \\ y = \rho\sin(\theta) \\ z = z \end{cases} \qquad \rho \in [0, +\infty), \ \theta \in [0, 2\pi], \ z \in \mathbb{R}$$

Si calcola

$$J = \rho \quad \Rightarrow \quad \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \ \to \ \rho\ \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$$

Quindi

$$\iiint_T f(x, y, z)\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_{\widetilde{T}} f(\rho\cos(\theta), \rho\sin(\theta), z)\rho\ \mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}z$$

- Passaggio alle coordinate cilindriche <u>conveniente</u> se

nell'espressione di $\underline{f}$, e/o nell'espressione di $\underline{T}$,

compaiono espressioni con

$$x^2 + y^2 \quad \text{(in coordinate cilindriche } \to \ \rho^2\text{)}$$

Es. 5.

$$I = \iiint_T (x^2 + z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

con

$$T = \left\{(x, y, z) \in \mathbb{R}^3 \; : \; \underline{x^2 + y^2} \leq z \leq 4\right\}.$$

In coord. cilindriche

$T \to \tilde{T}$: $\rho^2 \leq z \leq 4$

$\Rightarrow$ · $0 \leq z \leq 4$

· $0 \leq \rho \leq \sqrt{z}$

· nessun vincolo su θ: $\theta \in [0, 2\pi]$

$$I = \iiint_{\tilde{T}} \Big(\underbrace{\rho^2\cos^2(\theta)}_{=x^2} + z\Big) \overbrace{\rho \, d\rho \, d\theta \, dz}^{= dx\,dy\,dz}$$

$$= \iint_{[0,2\pi]\times[0,4]} \left(\int_0^{\sqrt{z}} (\rho^2\cos^2\theta + z)\,\rho\, d\rho \right) d\theta\, dz$$

$\tilde{T}$ è dominio normale rispetto al piano θz

$$= I_1 + I_2$$

$$I_1 = \iint_{[0,2\pi]\times[0,4]} \left(\int_0^{\sqrt{z}} \rho^3 \cos^2\theta \, d\rho \right) d\theta\, dz$$

$$= \iint_{[0,2\pi]\times[0,4]} \cos^2(\theta) \cdot \frac{1}{4} z^2 \, d\theta\, dz$$

$$= \frac{1}{4} \int_0^{2\pi} \cos^2(\theta)\,d\theta \int_0^4 z^2\,dz$$

$$= \frac{16}{3}\pi$$

$$I_2 = \iint_{[0,2\pi]\times[0,4]} \left(\int_0^{\sqrt{z}} \rho\, z \, d\rho \right) d\theta\, dz$$

$$= \iint_{[0,2\pi]\times[0,4]} z \cdot \frac{z}{2}\, d\theta\, dz$$

$$= \frac{64}{3}\pi$$

$$\Rightarrow I = \frac{80\pi}{3}$$

Es. 6.

$$I = \iiint_T \frac{1}{(x^2+y^2+z^2)^{1/2}} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

ove T

$$T = \left\{(x,y,z) \in \mathbb{R}^3 \, : \, z^2 \leq x^2+y^2 \leq 2z - z^2\right\}.$$

Si noti che

$$x^2+y^2 \leq 2z - z^2 \;\Leftrightarrow\; x^2+y^2+(z-1)^2 \leq 1$$

Quindi T è l'intersezione di

$$\begin{cases} x^2+y^2 \geq z^2 & \text{complement. di } x^2+y^2 < z^2 \\ x^2+y^2 \leq 2z - z^2 & \text{sfera con } C=(0,0,1) \; r=1 \end{cases}$$

• La presenza di x^2+y^2 nella def. di T suggerisce le coordinate cilindriche.

$T \to \widetilde{T}$

$$z^2 \leq x^2 + y^2 \leq 2z - z^2$$

$$\Updownarrow$$

$$z^2 \leq \rho^2 \leq 2z - z^2$$

$\longrightarrow$ in particolare : $z^2 \leq 2z - z^2$

$$\Downarrow$$

$$z(z-1) \leq 0 \iff z \in [0,1]$$

$$\Rightarrow \widetilde{T} : \begin{cases} z \in [0,1] \\ z \leq \rho \leq \sqrt{2z - z^2} \\ \theta \in [0,1] \end{cases}$$

dominio normale risp. piano θz

$$I = \iiint_{\tilde{T}} \frac{1}{(\rho^2+z^2)^{1/2}}\, \rho \, d\rho \, d\vartheta \, dz$$

integro per fili

$$= \iint_{[0,2\pi]\times[0,1]} \left(\int_z^{\sqrt{2z-z^2}} \frac{1}{(\rho^2+z^2)^{1/2}} \rho \, d\rho \right) d\vartheta \, dz$$

$$= \iint_{[0,2\pi]\times[0,1]} \Big[(\rho^2+z^2)^{1/2} \Big]_z^{\sqrt{2z-z^2}} d\vartheta \, dz$$

$$= \iint_{[0,2\pi]\times[0,1]} \left(\sqrt{2z} - \sqrt{2}\cdot z \right) d\vartheta \, dz$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right) \sqrt{2} \int_0^1 (\sqrt{z} - z)\, dz$$

$$= \frac{\sqrt{2}}{3}\pi$$

Es. 7.

$$I = \iiint_T z \sin(x^2 + y^2)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

con

$$T = \left\{(x, y, z) \in \mathbb{R}^3 \ : \ h \le z \le \sqrt{r^2 - x^2 - y^2}\right\}, \quad h > 0$$

$r > h$

$(0 \le)\, z \le \sqrt{r^2 - x^2 - y^2} \iff x^2 + y^2 + z^2 \le r^2$

$=$ sfera solida di centro $(0,0,0)$ e raggio r

· $z \ge h$: regione sopra il piano $z = h$

- Ragionevole usare le coordinate cilindriche, anche perché $x^2 + y^2$ è nella fz. integranda!

$T \to \widetilde{T}$

$$- \quad h \le z \le \sqrt{r^2 - \rho^2}$$

$$\Downarrow$$

$$- \quad h \le \sqrt{r^2 - \rho^2} \iff \rho^2 \le r^2 - h^2$$

$$\Updownarrow$$

$$\rho \in [0, \sqrt{r^2 - h^2}]$$

- su θ non ho vincoli

$\widetilde{T}$ è dominio "normale risp. piano $\rho\theta$":

$$\rho \in [0, \sqrt{r^2 - h^2}], \quad \theta \in [0, 2\pi], \quad h \le z \le \sqrt{r^2 - \rho^2}$$

Pongo $D = [0, \sqrt{r^2 - h^2}] \times [0, 2\pi]$

$$\Rightarrow I = \iint_D \left(\int_h^{\sqrt{r^2-\rho^2}} z \underbrace{\sin(\rho^2)\,\rho}\, dz \right) d\rho\, d\theta$$

$$= \iint_D \rho \sin(\rho^2) \left[\frac{z^2}{2} \right]_h^{\sqrt{r^2-\rho^2}} d\rho\, d\theta$$

$$= \frac{1}{2} \left(\int_0^{\sqrt{r^2-h^2}} \rho \sin(\rho^2) (r^2 - \rho^2 - h^2)\, d\rho \right) \times$$

$$\times \left(\int_0^{2\pi} 1\, d\theta \right)$$

$= \dots\dots$

$$= \frac{\pi}{2}\left[r^2 - h^2 - \sin\left(r^2 - h^2\right)\right]$$

Es. 8.

Calcolare il volume di T delimitato dalla superficie cilindrica

$$\mathcal{S}: \; x^2 + y^2 - 2x = 0$$

e dai piani $y = 0$, $z = 0$ e $z = \sqrt{2}$, e situato nel primo ottante.

· regione racchiusa da S:

$$x^2 + y^2 - 2x \leq 0$$

· 1° ottante: $x \geq 0, y \geq 0, z \geq 0$

Quindi

$$T: \begin{cases} x^2 + y^2 \leq 2x \\ x \geq 0, \; y \geq 0, \; 0 \leq z \leq \sqrt{2} \end{cases}$$

$T \to \widetilde{T}$

- $\rho^2 \leq 2\rho\cos\vartheta \quad (\star)$
- $\boxed{0 \leq z \leq \sqrt{2}}$
- $x \geq 0,\ y \geq 0 \Leftrightarrow \boxed{\vartheta \in [0, \pi/2]}$

da $(\star)$ deduco che

$$\rho(\rho - 2\cos(\vartheta)) \leq 0$$

$$\Leftrightarrow \boxed{\rho \leq 2\cos(\vartheta)} \quad (\text{essendo } \rho \geq 0)$$

$\Rightarrow$ è dominio normale rispetto a piano ϑz

$$I = \mathrm{vol}(T) = \iiint_T 1 \, dx\, dy\, dz$$

$$= \iiint_{\tilde{T}} \rho \, d\rho \, d\theta \, dz$$

$$= \iint_{[0,\pi/2]\times[0,\sqrt{2}]} \left(\int_0^{2\cos(\theta)} \rho \, d\rho \right) d\theta \, dz$$

$$= \iint_{[0,\pi/2]\times[0,\sqrt{2}]} 2\cos^2(\theta) \, d\theta \, dz$$

$$= 2\left[\int_0^{\pi/2} \cos^2(\theta)\, d\theta\right] \cdot \left[\int_0^{\sqrt{2}} dz\right] = \frac{\sqrt{2}}{2}\pi$$

Es. 9.

$$\iiint_T x^2 \, dx \, dy \, dz$$

con

$$T = \{(x,y,z) \in \mathbb{R}^3 : z \geq 0, \ x^2 + y^2 + 1 \leq z^2 \leq 4\}$$

$\tilde{T}$:

$\rho^2 + 1 \leq z^2 \iff \rho^2 \leq z^2 - 1$

$\Updownarrow$

$0 \leq \rho \leq \sqrt{z^2 - 1}$

Vincoli su z:

- $z^2 \leq 4$
- $z^2 \geq \rho^2 + 1 \geq 1$

$\Rightarrow 1 \leq z^2 \leq 4 \Rightarrow z \in [1, 2]$

(infatti $z \geq 0$)

non ci sono vincoli su ϑ: $\vartheta \in [0, 2\pi]$

$\Rightarrow \tilde{T}$ è un dominio "normale rispetto al piano ϑz"

$\vartheta \in [0, 2\pi]$, $\quad z \in [1, 2]$,

$0 \leq \rho \leq \sqrt{z^2 - 1}$

$$I = \iiint_{\tilde{T}} \rho^2 \cos^2\vartheta \, \rho \, d\rho \, d\vartheta \, dz$$

$$= \iint_{[0,2\pi]\times[1,2]} \left(\int_0^{\sqrt{z^2-1}} \rho^3 \cos^2\vartheta \, d\rho \right) d\vartheta \, dz$$

$$= \iint_{[0,2\pi]\times[1,2]} \cos^2(\theta) \left[\frac{\rho^4}{4}\right]_0^{\sqrt{z^2-1}} d\theta\, dz$$

$$= \iint_{[0,2\pi]\times[1,2]} \frac{1}{4} (z^2-1)^2 \cos^2(\theta)\, d\theta\, dz$$

$$= \frac{1}{4} \left(\int_0^{2\pi} \cos^2(\theta)\, d\theta\right) \cdot \left(\int_1^2 (z^4+1-2z^2)\, dz\right)$$

$$= \frac{19}{30}\pi$$